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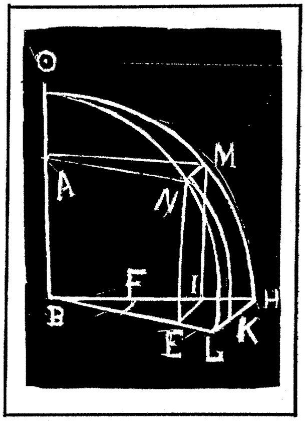
273. **Proposed by A. H. HOLMES, Brunswick, Maine.**

Required a purely geometrical solution of the problem, to find the contents of a solid generated by the revolution of a semi-segment of a circle about the sine of its arc.

Solution by the PROPOSER.

HBO is a quadrant whose revolution about *BO* as an axis generates a hemisphere. *BAF* is a semi-segment of radius$=BO$. Draw *AM* parallel to *HB* and *MI* parallel to *BO*. Suppose the quadrant to revolve about its axis a very small distance, the point *H* moving to *L* so as to generate *HBOLB*, *M* falling on *N*. Through *NE* pass a plane parallel to *HBO*. The semi-segment $HMI=AFB$ generates *HIELMN*; of which the part $EKLN=$part generated by *BAF*.



It is obvious that the volume generated by the semi-segment *BFA* in an entire revolution will equal that generated by *HMI* *minus* the sum of the solids *HIKEMN* lying about the circumference of the base of the hemisphere.

But $MNHIEK=IE\times$area of the semi-segment and the entire sum of all these is equal to the circumference described by *BI* as radius into the same area. If we put $BO=r$, $BE=c$, $BA=s$, and arc $AF=a$, we obtain for the solid generated by *BOMI*, $\frac{2\pi}{3}(sc^2+r^3-r^2s)$. Consequently,, for the solid generated by *HIM*, $\frac{2\pi}{3}(r^2s-sc^2)$. The sum of all the solids $HKEINM=$semi-segment $MHI\times2\pi c$ $=\pi(car-sc^2)$. Consequently the volume sought is

$$=\frac{2\pi}{3}(r^2s-sc^2)-\pi(car-sc^2)=\pi(\frac{sc^2}{3}+\frac{2r^2s}{3}-car).$$

Putting $c^2=r^2-s^2$, this becomes $\pi(sr^2-s^3/3-car)$.

GROUP THEORY.

14. **Proposed by O. E. GLENN, Springfield, Mo.**

Hölder has proved* that any group (G) of order $\sum_{i=1}^{n} p_i$ (p_i a prime$\neq p_j$) may be generated as follows: $M^\mu=N^\nu=1$, $N^{-1}MN=M^\alpha$, where $\{M\}$ is the product of all the invariant subgroups of G of prime order and $\{N\}$ is any one of a set of conjugate cyclical subgroups of order ν, ($\sum_{i=1}^{n} p_i=\mu\nu$). Find the generating relations of G in terms of operations of prime order, and express M and N in terms of these operations, for $n=4$.

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*See Burnside, *Theory of Groups*, p. 353.

Solution by the PROPOSER.

The group (G) contains a subgroup H_{pqr} of order pqr self-conjugately. This subgroup may be any one of the six known types

(1) $\{P\}\{Q\}\{R\}$, (2) $\{P\}\{Q, R\}$, (3) $\{Q\}\{P, R\}$, (4) $\{R\}\{Q, P\}$,
(5) $\{P, QR\}=[P^p=Q^q=R^r=1,\ QR=RQ,\ R^{-1}PR=P^\beta,\ Q^{-1}PQ=P^\alpha]$,
(6) $\{P, Q, R\}=[P^p=Q^q=R^r=1,\ QP=QP,\ R^{-1}PR=P^\delta,\ R^{-1}QR=Q^\gamma]$,

where by $\{\Omega\}$ is understood the cyclic group of order ω and $\{A, B\}=[A^a=B^b=1,\ B^{-1}AB=A^\theta]$. $\theta^b\equiv 1 \pmod{a}$.

Let N_t be the number of subgroups of order t in H. Then if

I) $N_p=1$, $N_q=1$, $N_r=1$, H is the type (1), and if S be any operation of G of order s we have $S^{-1}PS=P^\epsilon$, $S^{-1}QS=Q^\kappa$, $S^{-1}RS=R^\lambda$ where ϵ, κ, λ, may be part or all unity or primitive roots of the congruences $x^s\equiv 1 \pmod{p, q, r}$. Classification according to these values gives the following distinct types of G additional to the one abelian type $[1, 1, 1, 1]$.

$$P^p=Q^q=R^r=S^s=1, \quad \epsilon^s\equiv 1 \pmod{p}, \quad \kappa^s\equiv 1 \pmod{q}, \quad \lambda^s\equiv 1 \pmod{r}.$$

[1] $PQ=QP,\quad PR=RP,\quad QR=RQ,\quad S^{-1}PS=P^\epsilon,\quad SQ=QS,\quad SR=RS.$
[2] $PQ=QP,\quad PR=RP,\quad QR=RQ,\quad PS=SP,\quad S^{-1}QS=S^\kappa,\quad SR=RS.$
[3] $PQ=QP,\quad PR=RP,\quad QR=RQ,\quad PS=SP,\quad SQ=QS,\quad S^{-1}RS=R^\lambda.$
[4] $PQ=QP,\quad PR=RP,\quad QR=RQ,\quad S^{-1}PS=P^\epsilon,\quad S^{-1}QS=Q^\kappa,\quad SR=RS.$
[5] $PQ=QP,\quad PR=RP,\quad QR=RQ,\quad S^{-1}PS=P^\epsilon,\quad SP=QS,\quad S^{-1}RS=R^\lambda.$
[6] $PQ=QP,\quad PR=RP,\quad QR=RQ,\quad PS=SP,\quad S^{-1}QS=Q^\kappa,\quad S^{-1}RS=R^\lambda.$
[7] $PQ=PQ,\quad PR=RP,\quad QR=RQ,\quad S^{-1}PS=P^\epsilon,\quad S^{-1}QS=Q^\kappa,\quad S^{-1}RS=R^\lambda.$

II) Suppose next that $N_p=1$, $N_q=1$, $N_r=q$, so that H is of type (2). By a proper choice of R we get $S^{-1}RS=R^\lambda$, and in virtue of the relations

$$(RS)^{-1}Q(RS)=Q^{\gamma\kappa}=(SR^\lambda)^{-1}Q(SR^\lambda)=Q^{\kappa\gamma^\lambda}$$

we have $\gamma\kappa(\gamma^{\lambda-1}-1)\equiv 0 \pmod{q}$, $\lambda\equiv 1 \pmod{r}$. The parameters ϵ and κ may assume the same series of values as in the previous case. The new types are

[8] $PQ=QP,\quad PR=RP,\quad R^{-1}QR=Q^\gamma,\quad PS=SP,\quad SQ=QS,\quad SR=RS.$
[9] $PQ=QP,\quad PR=RP,\quad R^{-1}QR=Q^\gamma,\quad S^{-1}PS=P^\epsilon,\quad SQ=QS,\quad SR=RS.$
[10] $PQ=QP,\quad PR=RP,\quad R^{-1}QR,Q^\gamma,\quad PS=SP,\quad S^{-1}QS=Q^\kappa,\quad SR=RS.$
[11] $PQ=QP,\quad PR=RP,\quad R^{-1}QR=Q^\gamma,\quad S^{-1}PS=P^\epsilon,\quad S^{-1}QS=Q^\kappa,\quad SR=RS.$

III) Let $N_p=1$, $N_q=1$, $N_r=p$. Then H is the type (3), and since at least one subgroup of order r is permutable with S, a proper choice of R gives $S^{-1}RS=R^\lambda$. Reasoning similar to above gives

$$\epsilon\delta(\delta^{\lambda-1}-1)\equiv 0 \pmod{p}, \quad \lambda\equiv 1 \pmod{r}.$$

We thus arrive at the types

[12] $PQ=PQ$, $R^{-1}PR=P^{\delta}$, $RQ=QR$, $PS=SP$, $SQ=QS$, $SR=RS$.
[13] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $RQ=QR$, $S^{-1}PS=P^{\epsilon}$, $SQ=QS$, $SR=RS$.
[14] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $RQ=QR$, $PS=SP$, $S^{-1}QS=Q^{\kappa}$, $SR=RS$.
[15] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $RQ=QR$, $S^{-1}PS=P^{\epsilon}$, $S^{-1}QS=Q^{\kappa}$, $SR=RS$.

IV) Next suppose $N_p=1$, $N_r=1$, $N_q=p$, so that H is the type (4). Since $\{R\}$ is invariant in H it is invariant in G, so that $S^{-1}RS=R^{\lambda}$. Also a proper choice of Q,gives $S^{-1}QS=S^{\kappa}$. Then evidently

$$\alpha\varepsilon(\alpha^{\kappa-1}-1)\equiv 0 \pmod p),\quad \kappa\equiv 1 \pmod q)$$

which gives the additional types

[16] $Q^{-1}PQ=P^{\alpha}$, $PR=RP$, $PQ=QP$, $PS=SP$, $SQ=QS$, $SR=RS$.
[17] $Q^{-1}PQ=P^{\alpha}$, $PR=RP$, $PQ=QP$, $S^{-1}PS=P^{\epsilon}$, $SQ=QS$, $SR=RS$.
[18] $Q^{-1}PQ=P^{\alpha}$, $PR=RP$, $PQ=QP$, $PS=SP$, $SQ=QS$, $S^{-1}RS=R^{\lambda}$.
[19] $Q^{-1}PQ=P^{\alpha}$, $PR=RP$, $PQ=QP$, $S^{-1}PS=P^{\epsilon}$, $SQ=QS$, $S^{-1}RS=R^{\lambda}$.

V) If $N_p=1$, $N_q=1$, $N_r=pq$; H of type (6), there is again a subgroup of order r of (G) permutable with an arbitrary S, $S^{-1}RS=\mathrm{R}^{\lambda}$. Hence

$$\epsilon\delta(\delta^{\lambda-1}-1)\equiv 0 \pmod p),\quad \gamma\kappa(\gamma^{\lambda-1}-1)\equiv 0 \pmod q),\quad \lambda\equiv 1 \pmod r).$$

Again there are four new types, viz.,

[20] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $R^{-1}QR=Q^{\gamma}$, $SP=PS$, $SQ=QS$, $SR=RS$.
[21] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $R^{-1}QR=Q^{\gamma}$, $S^{-1}PS=P^{\epsilon}$, $SQ=QS$, $SR=RS$.
[22] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $R^{-1}QR=Q^{\gamma}$, $SP=PS$, $S^{-1}QS=Q^{\kappa}$, $SR=RS$.
[23] $PQ=QP$, $R^{-1}PR=P^{\delta}$, $R^{-1}QR=Q^{\gamma}$, $S^{-1}PS=P^{\epsilon}$, $S^{-1}QS=Q^{\kappa}$, $SR=RS$.

VI) Lastly, if $N_p=1$, $N_q=p$, $N_r=p$, [H of type (5)] we may by properly choosing R assume $S^{-1}RS=R^{\lambda}$. The generator of order q is not so at our choice but we may write

$$S^{-1}QS=P^{-v}Q^{\kappa}P^{v},\quad S^{-1}PS=P^{\epsilon}.$$

It follows as formerly that $\lambda\equiv 1 \pmod r)$. Also

$$(RS)^{-1}Q(RS)=P^{-v}Q^{\kappa}P^{v}=(SR)^{-1}Q(\mathrm{SR})=P^{-v\delta}Q^{\kappa}P^{v\delta}.$$

Hence $P^{-v(a^{\kappa}-1)}=P^{-v\delta(a^{\kappa}--1)}$, $v(a^{\kappa}-1)(\delta-1\equiv 0(\bmod p)$, and since $\kappa\not\equiv 0$, and $a\not\equiv 1$, $\delta\not\equiv 1$, we have $v\equiv 0 \pmod p)$ and

$(Q\mathrm{S})^{-1}P(Q\mathrm{S})=P^{\alpha\epsilon}=(\mathrm{S}Q^{\kappa})P(\mathrm{S}Q^{\kappa})=P^{\epsilon v^{\kappa}}$, so that $\kappa\equiv 1 \pmod q)$.

Combination of the remaining relations in all possible ways reveals no inconsistencies. We thus have the fresh types

[24] $Q^{-1}PQ=P^{\alpha}$, $R^{-1}PR=P^{\delta}$, $QR=RQ$, $SP=PS$, $SQ=QS$, $SR=RS$.
[25] $Q^{-1}PQ=P^{\alpha}$, $R^{-1}PR=P^{\delta}$, $QR=RQ$, $S^{-1}PS=P^{\epsilon}$, $SQ=QS$, $SR=RS$.

We next let T represent the number of distinct types contained in a given set of defining relations and tabulate T, M, N, and the relations connecting p, q, r, s for each of the 26 distinct types.

Type	T	*Arithmetical relations*	M	N
[1, 1, 1, 1]	1		$PQRS$	I
1	1	$p\equiv 1(s)$*	PQR	S
2	1	$q\equiv 1(s)$	PQR	S
3	1	$r\equiv 1(s)$	PQR	S
4	$s-1$	$p\equiv q\equiv 1(s)$	PQR	S
5	$s-1$	$p\equiv r\equiv 1(s)$	PQR	S
6	$s-1$	$q\equiv r\equiv 1(s)$	PQR	S
7	$(s-1)^2$	$p\equiv q\equiv r\equiv 1(s)$	PQR	S
8	1	$q\equiv 1(r)$	PQS	R
9	1	$p\equiv 1(s)$, $q\equiv 1(r)$	PQ	RS
10	1	$q\equiv 1(rs)$	PQ	RS
11	$s-1$	$p\equiv 1(s)$, $q\equiv 1(rs)$	PQ	RS
12	1	$p\equiv 1(r)$	PQS	R
13	1	$p\equiv 1(rs)$	PQ	RS
14	1	$p\equiv 1(r)$, $q\equiv 1(s)$	PQ	RS
15	$s-1$	$p\equiv 1(rs)$, $q\equiv 1(s)$	PQ	RS
16	1	$p\equiv 1(q)$	PRS	Q
17	1	$p\equiv 1(qs)$	PR	QS
18	1	$p\equiv 1(q)$, $r\equiv 1(s)$	PR	QS
19	$s-1$	$p\equiv r\equiv 1(s)$, $p\equiv 1(q)$	PR	QS
20	$s-1$	$p\equiv q\equiv 1(r)$	PQS	R
21	$r-1$	$p\equiv q\equiv 1(r)$, $p\equiv 1(s)$	PQ	RS
22	$r-1$	$p\equiv q\equiv 1(r)$, $q\equiv 1(s)$	PQ	RS
23	$(r-1)(s-1)$	$p\equiv q\equiv 1(rs)$	PQ	RS
24	1	$p\equiv 1(qr)$	PS	QR
25	1	$p\equiv 1(qrs)$	P	QRS

The method of determining T for a given type may be illustrated by type (23). Let $\Omega=S^xR^yQ^zP^w$ be the general operator of G. Then $\Omega^{\omega}=S^{\omega x}\,R^{\omega y}\,Q^{z\left(\frac{\kappa^{\omega x}\,\gamma^{\omega y}-1}{\kappa^x\gamma^{\omega}-1}\right)}P^{w\left(\frac{\varepsilon^{\omega x}\delta^{\omega y}-1}{\varepsilon^x\delta^y-1}\right)}$, and it follows that Ω is of order s only when y, z, and w are zero, of order r only when x, z, and w are zero, p only when x, y, z are zero, and q only when x, y, w are zero. Hence the most general transformation of G is given by $S_0=S^x$, $R_0=R^y$, $Q_0=Q$, $P_0=P$. These reproduce the type with δ, γ, ε, κ replaced by δ^x, γ^x, ϵ^y, κ^y. We may therefore fix δ and ϵ and replace γ and κ by γ^k, κ^h, respectively ($k=1, 2, \ldots, \overline{r-1}$, $h=1, 2, \ldots, \overline{s-1}$). Hence there are $(r-1)$ $(s-1)$ types of G.

The lowest order of the form $p\ q\ r\ s$ is 210. There are 11 groups of this order.

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*An abbreviation of $p\equiv 1$ (mod s).